Mean Field Dynamics in Non-Abelian Plasmas from Classical Transport Theory

The dynamics of mean fields in non-Abelian plasma is on the basis of mean fields and fluctuations. We present a general set of covariant equations describing the dynamics of classical transport theory, we present a general set of covariant equations describing the dynamics of classical transport theory. We present a general set of covariant equations describing the dynamics of classical transport theory.

The starting point is to consider gluons or electrons. Here, we have to consider gluons or electrons. Here, we have to consider gluons or electrons. Here, we have to consider gluons or electrons.

\[ \mathbf{J}(x) = \sum_{i} q_i \mathbf{e}_i \nabla \times \mathbf{A}(x) \]

\[ \mathbf{E}(x) = \frac{\partial \mathbf{A}(x)}{\partial t} - \mathbf{J}(x) \]

\[ \mathbf{B}(x) = \nabla \times \mathbf{A}(x) \]

The energy densities required to form a quark-gluon plasma might be taken as the input, to describe the quark-gluon plasma. The energy densities required to form a quark-gluon plasma might be taken as the input, to describe the quark-gluon plasma. The energy densities required to form a quark-gluon plasma might be taken as the input, to describe the quark-gluon plasma.

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the sequel, not be given explicitly.) Physical constraints are enforced through the phase space volume element \(d^3p \equiv 2\pi \rho(p) d^3p = \rho(p) \delta(p^2 - m^2)/(2\pi^3)\), while \(dQ\) contains delta functions imposing the group Casimirs (see [5] for their definition). The covariant conservation of the current \(\langle c \rangle\) is shown using \(\langle a \rangle\) [5].

If the system under study contains a large number of particles it is impossible to follow their individual trajectories in phase space. Thus, \(\langle a \rangle\) can no longer be considered a deterministic quantity and one has to switch to a statistical description, taking statistical averages \(\langle j \rangle\) of all microscopic quantities. We write

\[
\begin{align*}
A^a_\mu &= \bar{A}^a_\mu + a^a_\mu, \quad f = \bar{f} + \delta f, \quad J^a_\mu &= \bar{J}^a_\mu + \delta J^a_\mu, \quad (3)
\end{align*}
\]

where the quantities with a bar denote the mean values, e.g. \(\bar{f} = \langle f \rangle\) and \(\bar{A} = \langle A \rangle\), while the mean values of fluctuations vanish, \(\langle \delta f \rangle = 0 \) and \(\langle a \rangle = 0\). We also split

\[
\begin{align*}
F^a_{\mu\nu} &= \bar{F}^a_{\mu\nu} + f^a_{\mu\nu}, \quad (4a)
F^a_{\mu\nu} &= (\bar{F}^a_{\mu\nu} + f^a_{\mu\nu}) + g f^{abc} d^a_{\mu} d^b_{\nu}, \quad (4b)
\end{align*}
\]

with \(\bar{D} \equiv \nabla \bar{A}\) and \(\bar{F} \equiv \nabla \bar{A}\). Note that the mean field strength \(F^a_{\mu\nu} = \bar{F}^a_{\mu\nu} + g f^{abc} (a^b_{\mu} b^c_{\nu})\) due to the non-Abelian nature of the theory.

Let us take a statistical average of (2) to find the kinetic equations for the mean values,

\[
\begin{align*}
p^\mu \left( \bar{D}_\mu - g Q_a f^{a}_{\mu\nu} \bar{g}_{\nu} \right) \bar{f} &= \langle \eta \rangle + \langle \xi \rangle, \quad (5a)
\bar{D}_\mu F^{\mu\nu} + \langle J_{\mu\nu} \rangle &= \mathcal{F}^\nu. \quad (5b)
\end{align*}
\]

In (5a) we used \(\delta f = g f^{abc}(Q_a A^b_\mu \bar{g}_{\nu}) f = D_\mu \bar{f}\). The functions \(\eta, \xi, \) and \(J_{\mu\nu}\), are of second and higher order in the fluctuations and read

\[
\begin{align*}
\eta &= g Q_a p^\mu \partial^\nu f^{a}_{\mu\nu} \delta f, \quad (6a)
\xi &= g p^\mu f^{abc} Q^c (\partial^\nu d^a_{\mu} \delta f + g a^b_{\mu} b^c_{\nu} \bar{f}), \quad (6b)
J_{\mu\nu}^{a,\nu} &= g f^{abc} \left[ \bar{D}^a_{\mu} a^b_{\nu} d^c_{\nu} + a^a_{\mu} a^b_{\nu} f^{b\nu}_{\mu\nu} \right]. \quad (6c)
\end{align*}
\]

The corresponding equations for the fluctuations are obtained by subtracting (5) from (2),

\[
\begin{align*}
p^\mu \left[ \bar{D}_\mu - g Q_a f^{a}_{\mu\nu} \bar{g}_{\nu} \right] \delta f &= g p^\mu a^a_{\mu} f^{abc} Q^c_\nu \bar{g} \bar{F}\nonumber
- g Q_a [\bar{D}_\mu a^b_{\mu} - D_\mu a^b_{\mu}] \delta f = \eta + \xi - \langle \eta \rangle - \langle \xi \rangle.
\end{align*}
\]

\[
\begin{align*}
\bar{D}^a_{\mu} a^b_{\nu} - \bar{D}^a_{\nu} \bar{D}^b_{\mu} &= + 2 g f^{abc} F^{\mu\nu}_{\rho} a^\rho_{\mu\nu} + J_{\mu\nu}^{\nu,\nu} - \langle J_{\mu\nu}^{\nu,\nu} \rangle = \delta J_{\mu\nu}^{a,b}. \quad (7a)
\end{align*}
\]

A number of comments are in order.

1. The equations (5) and (7) are exact, no approximations have been made. In particular, they are also valid in out-of-equilibrium situations.

2. The equations (5) and (7) are consistent with gauge invariance. They are covariant under the mean gauge field symmetry \(\delta \bar{A}^a_\mu = (\bar{D}_\mu a^a_\nu) \delta \), and \(\delta a^a_\mu = g f^{abc} d^a_\mu d^b_\nu\), in analogy to the background field formalism [10]. This establishes the compatibility of the statistical average with the gauge transformations of the mean field. We postpone a careful and detailed discussion to [7].

3. The microscopic current conservation implies

\[
\begin{align*}
\left( \bar{D}_\mu f^\nu \right)_a + g f^{abc} \left( a^a_\mu \delta J^c_{\mu\nu} \right)_a &\equiv 0. \quad (8)
\end{align*}
\]

This is automatically consistent with (5b), provided \(\bar{J}\) and \(\delta J\) are solutions of (5) and (7). A similar equation holds for the fluctuation fields. Note that the validity of (8) turns into a non-trivial consistency check for approximate solutions.

4. The functions \(\langle \eta \rangle\) and \(\langle \xi \rangle\) can be considered as the effective collision integrals of the Boltzmann equation (5a). In our formalism the collision integrals arise as correlators of statistical fluctuations. The fluctuations of the gauge fields cause random changes in the motion of particles, and thus, they can be viewed as having the same effects as collisions. This can be seen as a derivation of collision integrals from the microscopic theory. Note also that the current induced by the fluctuations of the gauge field \(\langle J_{\mu\nu} \rangle\) is a purely non-Abelian effect.

5. A general procedure for integrating-out the fluctuations amounts to first solve their dynamics (7) in the background of mean fields. In general, this is a difficult task, in particular due to the non-linear terms in (7). The (explicit) solution is then inserted into (6). The back-coupling of the fluctuations to the mean fields is finally obtained after taking the statistical average of the functions (6a) and yields the effective collision integrals and the induced current in (5).

6. The set of equations (5) and (7) reproduces the known set of kinetic equations for Abelian plasmas in the corresponding limit [8], in which only the collision integral \(\langle \eta \rangle\) survives. The Abelian counterpart of \(\langle \eta \rangle\) can be expressed as the Balescu-Lenard collision integral [8]. One can then prove in a rigorous way the correspondence between fluctuations and collisions in the Abelian plasmas mentioned above. (An analogous derivation of collision integrals for Wigner functions can be found in [11], see also [12].)

7. Neglecting all fluctuations reduces (5) to the well-known (non-Abelian) Vlasov equations. This terminates the derivation and discussion of the basic set of equations.

To put the method to work we will specialize our analysis to hot non-Abelian plasmas close to equilibrium, with the gauge coupling \(g \ll 1\). This allows us to perform several approximations. We will consider small fluctuations, neglecting in (6a) and (6c) the terms cubic in the fluctuations. This is interpreted as neglecting effective three-body collisions versus binary ones. In the same spirit, we employ the second moment approximation for the dynamics of the fluctuations [8], setting \(\eta = \langle \eta \rangle, \xi = \langle \xi \rangle\) and \(J_{\mu\nu} = \langle J_{\mu\nu} \rangle\) in (7). This linearizes the dynamics of the fluctuations and can be interpreted as neglecting the influence of collisions on the dynamics of the fluctuations.
Finally, the term containing the mean field strength in (7a) is negligible compared to the remaining terms and will be omitted, as long as $g | \bar{F}_{\mu \nu}^{\text{eq}} | / m_D \ll T$, with $m_D$ the Debye mass [8].

We study the mean distribution function $\tilde{f}(x, p, Q) = \tilde{f}_{\text{eq}}(p) + g \tilde{f}^{(1)}(x, p, Q)$. In the strictly classical approach, the relativistic Maxwell distribution at equilibrium is used for any species of particles. Here, we consider only massless particles in the adjoint, with $\int dQ Q_s Q_b = N \delta_{s b}$. For particles in the fundamental one has $\int dQ Q_s Q_b = \frac{1}{2} \delta_{s b}$ instead. Solving (5) for vanishing fluctuations in this approximation gives the infinite set of non-Abelian hard thermal loops [5,6].

We now include small statistical fluctuations $\delta f$ around $\tilde{f}$ and re-write the approximations to (5) and (7) in terms of current densities and their fluctuations. Consider the current densities

\begin{align}
J_{\alpha_1 \ldots \alpha_q}^{\mu}(x, p) &= \xi p^\mu \int dQ Q_{\alpha_1} \ldots Q_{\alpha_q} \tilde{f}(x, p, Q), \\
\mathcal{J}_{\mu}(x, v) &= \int dP J_{\alpha_1 \ldots \alpha_q}^{\mu}(x, p).
\end{align}

The measure $dP$ only integrates over the radial components, $dP = dP_{\text{rad}}/4\pi$, and $\xi^\mu = (1, v)$ with $\xi^3 = 1$. The current (2c) is obtained performing the remaining angle integration $\int dQ \mathcal{J}(x, v)$. From now on we will omit the arguments of the current density $\mathcal{J}$, unless necessary to avoid confusion.

After multiplying (5a) by $g Q_a \xi^\mu / p_0$, summing over the two helicities, and integrating over $dPdQ$, we obtain for the mean current density at leading order in $g$

\begin{align}
\xi^\mu \bar{D}_\mu \mathcal{J}^\nu + m_D^2 v^\mu \xi^\nu \bar{P}_{\alpha 0} &= \langle \eta^\nu \rangle + \langle \xi^\nu \rangle, \\
\bar{D}_\mu \bar{P}^{\mu \nu} + \langle \eta_{\text{res}}^\nu \rangle &= \mathcal{P}^\nu,
\end{align}

with the Debye mass $m_D^2 = -2g^2 N \int dP_0 \int dP_{\text{eq}}(p_0) / dp_0$, and

\begin{align}
\eta^\mu &= -g \int \frac{dP}{p_0} \{ \bar{D}_\mu a^\nu - \bar{D}_\nu a^\mu \} \delta J_{\nu \mu}^{ab}(x, p), \\
\xi^\mu &= -g f_{abc} v^\rho \bar{a}_0^\rho \delta J_{\nu ab}^{\alpha c}(x, p),
\end{align}

\begin{align}
J_{\nu ab}^{\alpha c} &= g f_{abc} \bar{D}_\mu a^\mu \eta^\nu - \bar{D}_\nu a^\mu \eta^\mu + \delta^{\nu \mu} \delta^{\alpha c} \delta J_{\nu \mu}^{ab}.\end{align}

For the fluctuations we find

\begin{align}
[v^\mu \bar{D}_\mu \mathcal{J}^\nu]_{ab} &= -m_D^2 v^\mu \bar{D}_\mu a^\nu - \bar{D}_\nu a^\mu \delta J_{\nu \mu}^{ab}, \\
[v^\mu \bar{D}_\mu \delta \mathcal{J}^\nu]_{ab} &= g v^\mu \bar{D}_\mu a^\nu \{ f_{abc} \delta b^c + f_{bac} \delta a^c \} \delta J_{\nu bc}^{cd}, \\
[\bar{D}_\mu a^\nu - \bar{D}_\nu a^\mu]_{ab} &= + g f_{abc} \bar{D}_\rho a^\rho a^\mu + \delta J_{\mu a}^{bc}.
\end{align}

We solve the equations for the fluctuations (12) with an initial boundary condition for $\delta f$, and $a_\mu(t = 0) = 0$. Exact solutions to (12a) and (12b) can be obtained [7].

The current fluctuation $\delta \mathcal{J}_\mu$ reads, for $x_3 \equiv x - v \tau$ and the parallel transporter $\tilde{U}_{ab}$, obeying $v^\nu \bar{D}_\nu \tilde{U}_{ab}(x, y) |_{x_3 = 0} = 0$. In order to solve (12c) for $a_\mu$, we make a double expansion in both $\tilde{A}$ and $\tilde{J}$, using $\tilde{U}_{ab} = \delta_{ab} + O(g \tilde{A})$. We denote by $\delta^{(n)}$ the term containing a total of $n$ powers in the mean fields $\tilde{A}$ and/or $\tilde{J}$.

For our purposes, it will be sufficient to consider the zeroth order term in $\tilde{A}$, and the zeroth and first order terms in $\tilde{J}$. Using the one-sided Fourier transform [8], we find

\begin{align}
\delta a_{\text{res}}(k) &= \frac{1}{-k^3 + 4\pi} \int \frac{dQ}{4\pi} \delta J_{\nu a}^{(0)}(t = 0, k, v), \\
\delta a_{\text{res}}(k) &= \frac{1}{-k^3 + 4\pi} \int \frac{dQ}{4\pi} (k, v) x \times
\int \frac{d^3q}{(2\pi)^3} \langle \xi^\nu \rangle \delta^3(Q - q)
\end{align}

in the gauge $k \cdot a = 0$. The function $\Pi_T(k)$ is the transverse polarization tensor of the plasma, $\Pi_T(k) = \delta_{ij} - k_i k_j / k^2$ the transverse projector, and $\delta^{[3]}(\mathbf{p} - \mathbf{p'}) \delta(Q - Q') \tilde{f}(p) + \mu_{kp'p'} QQ$. The function $\delta(Q - Q')$ is, apart from a (representation dependent) normalization constant, a $N^2 - N$ dimensional $\delta$-function over the proper set of Darboux variables related to the color charges [5]. The second term in (15) is the Fourier transform of a smooth function that vanishes at large distances. The above statistical average is all we need to evaluate the collision integrals.

For the remaining part we will concentrate on the dynamics of mean fields with typical momenta around $g m_D$. When computing the related collision integrals, we will find logarithmic divergences, cut-off in the infrared by the inverse collision time. We employ the leading logarithmic approximation, assuming $\ln(1/g) \gg 1$ while neglecting all sub-leading (though finite) terms.

We find that the induced current $\langle J_{\nu a}^{(1)} \rangle$ vanishes, as do the fluctuation integrals $\langle \eta_{\text{res}} \rangle$ and $\langle \xi_{\text{res}} \rangle$. The vanishing of $\langle \eta_{\text{res}} \rangle$ is consistent with the fact that in the Abelian limit the counterpart of $\langle \eta \rangle$ vanishes at equilibrium [8].
In the same spirit we evaluate the collision integrals containing one \( \mathcal{J} \) field. Consider

\[
\left\langle \xi_{\mu,\nu}^{(1)} \right\rangle = g \int d\nu e^{-\nu} \left\{ -a_{\mu,\nu}^{(1)}(x) \delta \mathcal{J}_{\rho,\sigma}^{(1)}(x, v) + g \int d\nu e^{-\nu} \int_0^\infty d\tau J_{\rho,\sigma}(x, \tau, v) \left( a_{\mu,\nu}^{(0)}(x) \mathcal{J}_{\rho,\sigma}^{(0)}(x, \nu) \right) \right\},
\]

which simplifies, at logarithmic accuracy, to

\[
\left\langle \xi_{\mu,\nu}^{(1)}(x, v) \right\rangle = -\frac{g^2}{4\pi} NT \ln \left( \frac{1}{y} \right) \times v_f \int d\Omega v' \mathcal{J}(v, v') J_{\mu,\nu}(x, v'),
\]

\[I(v, v') \equiv \delta^{(2)}(v - v') - \frac{4}{\pi} \frac{(v - v')^2}{\sqrt{1 - (v - v')^2}},\]

The above expression has been obtained first in [3], and reproduces the collision integral considered in the Boltzmann equation of [13].

We verified that the leading logarithmic solution is consistent with gauge invariance. Evaluating the correlator in (8) yields \( \mathcal{J}^\mu = 0 \), in accordance with (10b) in the present approximation.

Following Bödeker, one can now estimate \( \mathcal{J}^\mu \) from (10a) to obtain for (10b)

\[
(\mathcal{D}^\mu \bar{F}^{\lambda \nu})_{\rho,\sigma} = \sigma \bar{F}^{\lambda \nu}_{\rho,\sigma} \nu^\mu, \quad \sigma = -\frac{4\pi m^2_D}{3Ng^2T \ln \left( \frac{1}{y} \right)}.
\]

This is the result of [3]. The coefficient \( \sigma \) represents the color conductivity and has been discussed in [13, 14]. The white noise \( \nu \) has its origin in the fluctuations of the transverse part of \( \xi^{(1)} \) [3, 7]. We obtain to leading order

\[
\left\langle \nu^\mu_a(x) \nu^\nu_b(y) \right\rangle = 2T \sigma \delta^a_b \delta_{\rho,\sigma} \delta^{(2)}(x - y),
\]

in accordance with the fluctuation-dissipation theorem (FLT). Note also that the classical Debye mass differs from the quantum one.

In order to go beyond classical transport theory we expand about the bosonic (fermionic) quantum-statistical equilibrium distribution function \( \bar{f}_{\mu,\nu}(x) \). For gluons in the adjoint, the Debye mass obtains as \( m_D^2 = g^2 NT^2/3 \). The FDT is obeyed as well, if \( \mathcal{J} \) in (15) is replaced by \( \bar{f}_{\mu,\nu} \) (1 \( \pm \bar{f}_{\mu,\nu} \)). This should however be derived in a way that is consistent with the microscopic theory [7]. Also, all the quantum integrals are obtained with the correct statistical factors [7]. It is interesting to note that all quantum modifications are contained in the implicit change of \( m_D^2 \).

This terminates the explicit derivation, in the leading logarithmic approximation, of the collision integral and the dynamical equations for the soft fields from classical transport theory.

Summarizing, we have given a prescription to derive mean gauge field equations from classical transport theory. This includes a recipe to obtain effective (classical or quantum) collision integrals from the microscopic theory. The approach is in accordance with gauge invariance. In a close-to-equilibrium plasma and for small gauge coupling, we reproduce Bödeker's effective theory.

The last part of our analysis can straightforwardly be generalized in order to obtain explicit expressions for the collision integrals not only for the soft momentum region. Another interesting open problem is using the same methods for out-of-equilibrium situations. Based on the evaluation of collision integrals for non-Abelian plasmas and equilibration [8], we should find the Coulomb logarithm changing drastically the mean non-Abelian gauge field equations.

It remains remarkable that classical transport theory is efficient enough as to describe not only the non-Abelian dynamics of semi-hard modes with momenta around \( m_D \), but as well the non-perturbative dynamics of soft gluons at leading logarithmic order. This establishes a link even beyond the one-loop level between our approach and a complete quantum field theoretical treatment, whose deeper structure is waiting for being uncovered [15].